Passive Scalar Transport by Traveling Wave Fields

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Abstract

We study passive scalar transport by a stochastic traveling wave field in the physically relevant limit where typical fluid velocities, U_o, are much smaller than the typical group velocity, co, of the waves. We derive the diffusion equation for the mean concentration field and compute the effective diffusion constant, D, as a power series in the small parameter u_0/c_0 for spatially smooth velocity fields. What is unusual about this problem is that for advection by wave fields of this type the zeroth order term vanishes, and the main contribution comes at order $(u_0/c_0)^2$. We use this to estimate D for long internal gravity waves in a shallow rotating fluid, and for wind generated surface gravity waves on deep water.

I. INTRODUCTION

Passive scalar transport by turbulent velocity fields has been a subject of interest to fluid dynamicists for many years (Taylor, 1921; Kraichnan, 1970; Gawedzki and Kupiainen, 1995; Chertkov and Falkovich, 1996; Chertkov, *et al., 1996).* The problem is of great importance in ocean and atmosphere dynamics where the transport of heat, moisture, salt and biogeochemical quantities has short term (weather) as well as long term (climate) implications. Theories of passive scalar transport to date have focused mainly on the effects of randomly moving eddies in vertical flows, as modelled, for example, by the Navier-Stokes equations in two and three dimensions. However vertical motions are not the only type of multiscale random velocity field capable of passive tracer transport. In the present work we consider an alternative mechanism, namely *wave-induced* diffusion, which has long been suspected to play a role in ocean dynamics. This class of random motions includes wind-generated surface gravity waves on deep water, Rossby and long internal gravity waves [known as baroclinic inertia-gravity waves] in rotating fluids, and other types of oscillating flows. These flows tend to be only weakly nonlinear, and are then characterized by a well defined dispersion law, $\omega(\mathbf{k})$, which concentrates their spectra on *surfaces* in frequency-wavenumber space. A crucial fact, as we shall see, is that the frequency spectra tend to have vanishing weight near zero frequency.

According to Richardson's empirical law (Richardson, 1926) [later derived by Batchelor (Batchelor, 1952) and supported by laboratory and field experiments in the ocean (Richardson and Stommel, 1948; Stommel, 1949; Monin and Özmidov, 1981)], the eddy diffusion coefficient D(L) decreases with decreasing eddy size, L, according to

$$D(L) = B\epsilon^{1/3}L^{4/3}, \tag{1}$$

where B is a constant of order unity and ϵ is the rate of energy transfer to large scales (induced by the usual inverse cascade in two dimensions). This equation is then valid for length scales larger than the energy input scale, L_{in}. For large scale oceanic motions, ϵ is of order $10^{-4}\mathrm{cm}^2/\mathrm{s}^3$. At scales below L_{in} the kinetic energy spectrum is controlled by the direct cascade of enstrophy (squared vorticity), and one has in place of (1)

$$D(L) = B'\epsilon_{\Omega}^{1/3}L^2, \tag{2}$$

where ϵ_{Ω} is the rate of enstophy transfer to small scales. Present eddy-resolving numerical models of ocean circulation can use spatial grid sizes as small as 10km. According to the Richardson law, *D(L)*, used in these models to account for "sub-grid" motions on scales smaller than L, should then be of order $10^6\mathrm{cm}^2/\mathrm{s}$. For comparison, the molecular diffusion coefficient for clean water is of order $10^{-5}\text{cm}^2/\text{s}$. One of the questions we address is, as L is pushed ever lower, would the actual diffusion eventually approach the molecular diffusion limit as eddies of all sizes are accounted for explicitly in an “ideal” numerical model? Or, are there other significant contributions to D that must be accounted for? As we show in the present work, wave induced diffusion, although usually small by comparison with eddy induced transport, may well set a lower bound on turbulent diffusion in natural environments.

Our main result, equation (28), provides a very general, quantitatively accurate estimate for the diffusion coefficient, valid for any linear or weakly nonlinear statistically isotropic wave field (generalization to anisotropic situations is straightforward). The theory is then applied to the case of oceanic baroclinic inertia-gravity waves (with length scales ranging from 10-1000km) and deep water surface gravity waves (with length scales ranging from 1-300m).

II. PASSIVE SCALAR DYNAMICS AND STATISTICS

Our analysis begins from the standard passive scalar transport equation

$$\frac{\partial q}{\partial t} + v \cdot \nabla q = \kappa \nabla^2 q, \tag{3}$$

where $q(x, t)$ is the passive scalar (salinity, heat, phytoplankton, etc.) concentration field, $v(x, t)$ is the advecting velocity field, which we take to be given with Gaussian statistics and homogeneous two-point correlator

$$C(x - x', t - t') = \langle \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}', t') \rangle, \tag{4}$$

and κ is the microscopic diffusion constant which we shall take to be zero since it is much smaller than the observed large scale diffusion constant due to v.

Since, in the absence of damping and strong nonlinear interactions between waves, travelling wave fields have a well defined dispersion law $\omega(\mathbf{k})$, the velocity field maybe decomposed in the form (for convenience the dimension, d, of space is kept general until the end)

$$\mathbf{v}(\mathbf{x},t) = \int \frac{d^dk}{(2\pi)^d}\left[\hat{\mathbf{v}}(\mathbf{k})e^{-i[\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t]} + c.c.\right] \tag{5}$$

The spatio-temporal spectrum is therefore restricted to certain *surfaces* in (k, ω) space. For example, for inertia-gravity waves one has $\omega(\mathbf{k}) = \sqrt{f^2 + (kc)^2}$, where the Coriolis parameter $f = 2\Omega\sin(\phi)$ (Ω being the earth's rotation frequency and #J being the latitude), and $C \simeq$ 2m/s is the phase speed of sufficiently short wavelength baroclinic inertia-gravity waves for which the Coriolis force is negligible. The amplitudes $\hat{\mathbf{v}}(\mathbf{k})$ are taken to be independent Gaussian random variables:

$$\langle\hat{\mathbf{v}}(\mathbf{k}) \cdot \hat{\mathbf{v}}^*(\mathbf{k}')\rangle = F(\mathbf{k})\delta(\mathbf{k} - \mathbf{k}') \tag{6}$$

where $F(k) \geq 0$ is real and physically is strongly peaked at some characteristic wavevector k_0, and decays fairly rapidly on either side. The Fourier transform of $C(\mathbf{x},t)$ is given by

$$\Phi(\mathbf{k}, \omega) = F(\mathbf{k})\delta[\omega - \omega(\mathbf{k})] + F(-\mathbf{k})\delta[\omega + \omega(-\mathbf{k})] \tag{7}$$

Nonlinear interactions between waves will broaden the delta functions here. One finds in the case of inertia gravity waves, for example, a broadening corresponding to a nonlinear decorrelation time of order 10-30 wave periods (Glazman, 1996 b). This will become important for passive scalar *correlations* at separations less than 10-30 wavelengths, but is not important for the diffusion constant.

Given the above characterization of the statistics of the velocity field, v, we would now like to compute the statistics of the passive scalar field, q. This paper will be concerned with deriving an equation of motion for the average, $\langle q(x, t)\rangle$. We shall see that under certain conditions a diffusion equation

$$\frac{\partial}{\partial t}\langle q(\mathbf{x},t)\rangle = D(t)\nabla^2\langle q(\mathbf{x},t)\rangle, \tag{8}$$

emerges. The main interest will be in estimates for $D_\infty = \lim_{t\to\infty} D(t)$.

III. RANDOM WALK REPRESENTATION

The computations are based on the following random walk representation for $q(\mathbf{x}, t)$ (Piterbarg, 1997):

$$q(\mathbf{x}, t) = q_0[\mathbf{Z}_{\mathbf{x},t}(0)], \tag{9}$$

in which go(x) = $q(\mathbf{x}, t$ = O) is the initial passive scalar distribution. The Lagrangian coordinate $\mathbf{Z}_{\mathbf{x},t}(s)$ is freely advected by the flow and satisfies

$$\frac{d\mathbf{Z}_{\mathbf{x},t}(s)}{ds} = \mathbf{v}(\mathbf{Z}_{\mathbf{x},t}(s), s), \tag{1o}$$

with the boundary condition that $\mathbf{Z}_{\mathbf{x},t}(t)$ = x. The subscripts on $\mathbf{Z}_{\mathbf{x},t}(s)$ therefore define the unique fluid particle that passes through the point x at time t. The point from which this particle started its motion at time zero is then $\mathbf{Z}_{\mathbf{x},t}(0)$, and at time s this same particle will be (or was) at point $\mathbf{Z}_{\mathbf{x},t}(s)$. The concentration at (x, t) is therefore determined by the initial concentration at the point at time zero which evolves into (x, t) under the flow. This equation may be written in the integral form

$$\mathbf{Z}_{\mathbf{x},t}(s) = \mathbf{x} + \int_t^s ds' \mathbf{v}(\mathbf{Z}_{\mathbf{x},t}(s'), s'). \tag{11}$$

Using the Fourier representation, *go(x)* = $\int d^d k (2\pi)^{-d} \hat{q}_0(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$, one then obtains

$$\langle q(\mathbf{x}, \ t)\rangle \ \int \frac{d^d k}{(2\pi)^d} \hat{q}_0(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x} - \lambda(\mathbf{k},t)} \tag{12}$$

in which

$$\begin{aligned} \lambda(\mathbf{k}, \ t) \ &= \ - \ \ln\langle e^{-i\mathbf{k}\cdot(\mathbf{Z}_{\mathbf{x},t}(0) - \mathbf{x})}\rangle \\ &= i\mathbf{k} \cdot \mathbf{x}_0(t) + \frac{1}{2} \sum_{ij} \sigma_{ij}(t) k_i k_j + O(k^3). \end{aligned} \tag{13}$$

Here $\mathbf{x}_0(t) = \langle \mathbf{Z}_{\mathbf{x},t}(0)$ – x) is a drift term. For unbiased flows, which we shall assume, this term vanishes. Subscripts i, j = 1,. ... d label the Cartesian components. The tensor σ_{ij} is given by

$$\sigma_{ij}(t) = \langle [\mathbf{Z}_{\mathbf{x},t}(0) - \mathbf{x}]_i [\mathbf{Z}_{\mathbf{x},t}(0) - \mathbf{x}]_j \rangle - x_{0,i}(t) x_{0,j}(t). \tag{14}$$

For isotropic flows one has $\sigma_{ij}(t) = \sigma(t)\delta_{ij}$, and $\sigma(t) = \frac{1}{d}\langle |\mathbf{Z}_{\mathbf{x},t}(0) - \mathbf{x}|^2 \rangle$. This diffusion term represents therefore the mean square distance travelled by a Lagrangian particle in time t, relative to any systematic drift. In general σ_{ij} is not simply a multiple of the identity matrix. However, it may always be diagonalized by an orthogonal transformation to yield a set of principal axes whose eigenvalues correspond to the (different) rates of diffusion along these axes. This general case has important geophysical applications, for example to flows in the β-plane (where latitudinal variations of the Coriolis force are taken into account), but we shall not pursue these here.

Equations (12), (13) and (14) immediately imply the validity of the diffusion equation (8). Direct substitution then yields [in the isotropic case with $\mathbf{x}_0(t) = 0$, and ignoring cubic and higher order terms in the k_i] the identification

$$D(t) = \frac{1}{2}\frac{d\sigma}{dt} = \frac{1}{d}\int_0^t G(s)ds \tag{15}$$

in which

$$G(s-t) = \langle \mathbf{v}(\mathbf{Z}_{\mathbf{x},t}(s), s) \cdot \mathbf{v}(\mathbf{x}, t) \rangle \tag{16}$$

is the Lagrangian temporal correlation function, and is an even function of its argument. Higher order terms in (13) represent corrections to diffusive behavior and become important on shorter length scales. If $q_0(\mathbf{x})$ varies only on large length scales, $\hat{q}_0(\mathbf{k})$ will vanish strongly for large k, and these terms may be ignored.

IV. SYSTEMATIC COMPUTATION OF DIFFUSION COEFFICIENTS

The aim now is to compute $D(t)$ systematically. The computation is based on the exact solution to the problem in which $\mathbf{v}(\mathbf{x}, t) \approx \mathbf{v}(0, t)$ is taken to be independent of the spatial coordinate. Perturbation theory will then be performed about this limit. In this limit one has $G(t) = G_0(t)$ where

$$G_0(s-t) = \langle \mathbf{v}(\mathbf{x},s) \cdot \mathbf{v}(\mathbf{x},t)\rangle = \int \frac{d\omega}{2\pi}\hat{S}(\omega)e^{i\omega(s-t)}, \tag{17}$$

where

$$\hat{S}(\omega) = \int \frac{d^d k}{(2\pi)^d}\Phi(\mathbf{k}, \omega) \tag{18}$$

is the frequency spectrum. Onetherefore obtains the Kubo-like formula (Kubo, 1963) for the lowest order result

$$D_\infty \approx D_\infty^{(0)} \equiv \frac{1}{d}\hat{S}(0). \tag{19}$$

Because $\mathbf{v}(0, t)$ is assumed Gaussian, all higher order terms in k in (13) in fact vanish in this limit.

Now, for the wave field problem of interest, we have the interesting result that $F(\mathbf{k}, \omega = 0) \approx 0$: the spectrum has negligible weight at zero frequency. Thus $D_\infty^{(0)} \simeq 0$, and the process appears subdiffusive. In order to obtain finite estimates, slow spatial variation in $\mathbf{v}(\mathbf{x}, t)$ must therefore be allowed for. We perform a Taylor expansion

$$\begin{aligned} \mathbf{v}(\mathbf{Z}_{\mathbf{x},t}(s),s) &= \mathbf{v}(\mathbf{x},s) + \{[\mathbf{Z}_{\mathbf{x},t}(s) - \mathbf{x}] \cdot \nabla\}\mathbf{v}(\mathbf{x},s) \\ &+ \frac{1}{2}\{[\mathbf{Z}_{\mathbf{x},t}(s) - \mathbf{x}] \cdot \nabla\}^2\mathbf{v}(\mathbf{x},s) + \ldots . \end{aligned} \tag{20}$$

This expansion is valid so long as $\mathbf{Z}_{\mathbf{x},t}(s) - x$ remains small compared to the typical length scale of variation of $\mathbf{v}(\mathbf{x}, t)$, i.e. the correlation radius, R_0, of $C(\mathbf{x}, 0)$. As seen above, time scales are governed by the decorrelation time, τ_0, of $G_0(t) = C(0, t)$. Defining the mean square velocity, $u_0^2 = \langle v^2\rangle$, the perturbation expansion is valid in the limit $u_0 \ll R_0/\tau_0 \approx c_0$, where c_0 is the typical group velocity of the waves. In other words, one requires that the typical particle velocity be much smaller than the typical wave velocity. ¿From (11) one has then

$$\begin{aligned} \mathbf{Z}_{\mathbf{x},t}(s) &= \mathbf{x} + \int_t^s ds'\mathbf{v}(\mathbf{x}, s') \\ &+ \int_t^s ds' \int_t^{s'} ds''[\mathbf{v}(\mathbf{x}, s'') \cdot \nabla]\mathbf{v}(\mathbf{x}, s') + \ldots, \end{aligned} \tag{21}$$

and therefore, substituting once more into (20),

$$\mathbf{v}(\mathbf{Z}_{\mathbf{x},t}(s), s) = \mathbf{v}(\mathbf{x}, s) + \int_t^s (is'[\mathbf{v}(\mathbf{x}, s') \cdot \nabla]\mathbf{v}(\mathbf{x}, s)$$
$$+ \int_t^s ds' \int_t^{s'} ds''[\mathbf{v}(\mathbf{x}, s'') \cdot \nabla][\mathbf{v}(\mathbf{x}, s') \cdot \nabla]\mathbf{v}(\mathbf{x}, s)$$
$$+ \ldots, \tag{22}$$

where the derivatives act on *all* spatial dependence to the right.

Equation (22) is now used to compute $G(t)$, with the result $G(t) = G_0(t) + G_2(t) + \ldots$. with $G_0(t)$ as above and

$$G,(s - t) = \int_t^s ds' \int_t^{s'} ds'' \langle \mathbf{v}(\mathbf{x}, t) . [\mathbf{v}(\mathbf{x}, s'') . \mathrm{V}]$$
$$\times [\mathbf{v}(\mathbf{x}, s') . \nabla]\mathbf{v}(\mathbf{x}, s)\rangle. \tag{23}$$

It has been assumed in this expansion that all odd moments of v vanish. Wick's theorem is now used to express the four point average in terms of the product of two point averages. The result is (we take here simply $\langle v_i v_j \rangle = \frac{1}{d}\delta_{ij}C$; incompressibility and anisotropy effects alter this, but can be handled straightforwardly)

$$G_2(t) = \frac{1}{d^2} \lim_{\mathbf{x}.\mathbf{x}' \to 0} [\nabla^2 + \mathrm{V} . \mathrm{V}'] \int_0^t ds \int_0^s ds'$$
$$\times [C(\mathbf{0}, s')C(\mathbf{x} - \mathbf{x}', t - s) + dC(\mathbf{x}, t)C(\mathbf{x}', s - s')$$
$$+ C(\mathbf{x}', s)C(\mathbf{x}, t - s')]$$
$$= \frac{1}{d^2} \int_0^t ds \int_0^s ds'[dC(\mathbf{0}, s - s')\nabla^2 C(\mathbf{0}, t)$$
$$+ C(0, s)\nabla^2 C(\mathbf{0}, t - s')], \tag{24}$$

where in the last equality it has been assumed that $C(\mathbf{x}, t)$ has a quadratic maximum at the spatial origin so that $\nabla C(\mathbf{0}, t) \equiv 0$ for all t. We may then immediately compute $D_\infty = D_\infty^{(0)} + D_\infty^{(2)} + \ldots$, with

$$D_\infty^{(2)} = \frac{1}{d} \int_0^\infty G_2(t)$$
$$= \frac{1}{d^3} \int_0^\infty dt \int_0^t ds \int_0^s ds'[dG_0(s - s')H_0(t)$$
$$+ G_0(s)\, H_0(t - S')], \tag{25}$$

where $G_0(t)$ and $H_0(t) \equiv \nabla^2 C(\mathbf{0}, t)$ are both even functions of time, with Fourier transforms $\hat{S}(\omega)$ and $\hat{T}(\omega)$. Let $A(t) = \int_t^\infty G_0(s)ds$ and $B(t) = \int_t^\infty H_0(s)ds$, then

$$D_\infty^{(2)} = \frac{1}{d^2} A(0) \int_0^\infty B(t)dt + \frac{1-d}{d^3} \int_0^\infty A(t)B(t)dt$$
$$= \int_{-\infty}^{\infty} \frac{d\omega}{4d^3\pi\omega^2} [\hat{T}(0) - \hat{T}(\omega)][\hat{S}(0) + (d-1)\hat{S}(\omega)]. \tag{26}$$

Using (7), and assuming an isotropic frequency spectrum, one has the simple relation

$$\hat{T}(\omega) = -\int \frac{d^d k}{(2\pi)^d} k^2 F(\mathbf{k}, \omega) = -k(\omega)^2 \hat{S}(\omega), \tag{27}$$

where $k(\omega)$ is the inverse of the function $\omega(k)$. For inertia gravity waves (see below) one has simply $k(\omega)^2 = (\omega^2 - f^2)/c^2$. Our final expression for $D_\infty^{(2)}$ in the case of interest where $\hat{S}(0) = 0$ is then

$$D_\infty^{(2)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{d-1}{2d^3} \frac{k(\omega)^2}{\omega^2} \hat{S}(\omega)^2. \tag{28}$$

To estimate orders of magnitude, it is useful to scale this expression using physical quantities. Let f_0 be a characteristic frequency of the system and let $x = \omega/f_0$ then be the dimensionless frequency. Since $u_0^2 = \int (d\omega/2\pi)\hat{S}(\omega)$, one may write

$$\hat{S}(\omega) = \frac{u_0^2}{f_0} s(x); \int_{-\infty}^{\infty} \frac{dx}{2\pi} s(x) = 1. \tag{29}$$

Similarly, one writes $k(\omega)^2/\omega^2 = K(x)^2/c^2$, where c is a characteristic phase or group velocity and $K(x)$ is dimensionless. For inertia gravity waves one has $K(x)^2 = 1 - x^{-2}$ (with the choice $f_0 = f$). The diffusion constant then may be expressed as

$$D_\infty^{(2)} = \frac{u_0^4}{f_0 c_0^2} B_d, \tag{30}$$

where all of the detailed spectral properties are characterized by the dimensionless quantity

$$B_d = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{d-1}{2d^3} K(x) s(x)^2. \tag{31}$$

Since s(z) has unit integral, roughly speaking B_d will be small compared to unity if $s(x)$ is very broad, and large compared to unity if it is strongly peaked.

Putting in some rough numbers for order of magnitude estimates, for $u_0 \sim$ 20cm/s, $c_0 \sim$ 2m/s and $1/f_0 \sim 1$ day one obtains $u_0^2/f_0 \sim$ 550m²/s. From (19), this would ordinarily be a rough estimate for the diffusion constant itself. However in our case the leading contribution is down by a factor $(u_0/c_0)^2 \sim 0.01$ from this, so that $u_0^4/f_0 c_0^2 \sim 5.5\mathrm{m}^2/\mathrm{s}$. Accurate estimates for D_∞ then rely on estimates for 11_2.

V. APPLICATIONS TO INERTIA-GRAVITY WAVES

Large scale oceanic and atmospheric motions (as occur in a thin layer of a rotating fluid) satisfy the hydrostatic approximation for the pressure field which then leads to a simplified set of equations known as the shallow water equations (LeBlond and Mysak, 1978; Gill, 1982). These equations contain, in addition to the usual horizontal vertical flows, in the linear approximation, a set of oscillating solutions known as inertia-gravity (IG) or Poincaré waves. While the time scale of the former is measured in weeks (being limited in principle only be the size of the ocean), the period of IG waves actually has a lower bound determined by the latitude. As described earlier, the dispersion relation in the f-plane approximation (neglecting spatial variation of the Coriolis parameter f) is given by

$$\omega^2 = f^2 + c^2 k^2, \tag{32}$$

where c is the (constant) wave phase speed in the absence of the earth's rotation (known as the Kelvin wave speed). Wave solutions with frequency smaller than f therefore do not exist. This turns out to be a slight oversimplification: see below. Of main interest in oceanography are IG waves occuring at the interface, known as the *thermocline depth*, between two horizontal layers with slightly different densities. These long internal waves, called baroclinic inertia-gravity (BIG) waves, account for most of the energy in oceanic motions with times scales less than one day. The corresponding amplitude of thermocline depth oscillations may attain several tens of meters while the horizontal velocity scale is about 10cm/s. Being weakly to moderately nonlinear, BIG waves are characterized by a broad frequency spectrum resulting from the Kolmogorov type cascades of wave energy and wave action (Falkovich, 1992; Falkovich and Medvedev, 1992; Glazman, 1996a; Glazman, 1996 b). Among known causes of BIG waves in the ocean are the scattering of semi-diurnal barotropic tides by topographic features on the ocean floor, fluctuations of wind stress and atmospheric pressure at the ocean surface, and amplification of internal waves by mesoscale-eddy fields and shear flows (Fabrikant, 1991; Stepanyants and Fabrikant, 1989; Troitskaya and Fabrikant, 1989).

An experimental frequency spectrum of horizontal velocity fluctuations in the upper ocean layer, based on data reported in (Fu, 1981), is illustrated in Fig. 1. Notice that the main spectral peak actually *spans* the Coriolis frequency, f (the second peak is due to the semi-diurnal tide). This means that the modes dominating the spectrum are actually of sufficiently long wavelength that the f-plane approximation is no longer valid: Evanescent tails of lower frequency waves that exist at slightly smaller latitudes actually contribute substantially to the spectrum. This has a very strong effect on estimates for the diffusion constant as the spectral factor $k(\omega)$ in (28) vanishes at f, thereby suppressing the integrand in the region near the peak. In future work we will account for this effect properly by using the β-plane approximation [which allows linear variation of $f(y) = f(y_0) + \beta(y - y_0)$ with latitude] to compute the dispersion relation. This will lead to a nonvanishing $k(\omega)$ near the peak, vastly increasing the result for $D_\infty^{(2)}$. Presently we can rigorously estimate only the contribution from the isotropic short-scale range of the IG wave spectrum for which the isotropic dispersion law (32) is valid. Only in this range is our main result (28) strictly valid. We therefore reduce the range of integration to the interval (ω_0, ∞) with $\omega_0 > f$. The spectrum plotted in Fig. 1 yields $D_\infty^{(2)}$ ranging from $70\mathrm{cm}^2/\mathrm{s}$ for $\omega_0 = 1.05f$ to $50\mathrm{cm}^2/\mathrm{s}$ for $\omega_0 = 1.5f$. As mentioned above, we anticipate that the actual $D_\infty^{(2)}$ will be much greater when the entire spectral peak is taken into account in the β-plane approximation. A crude estimate of the effect may be obtained by replacing $k(\omega)/\omega$ by $\max\{k(\omega)/\omega, 1/c\}$ in (28) so that this coefficient retains the finite value 1/c near f. Taking c to be the Kelvin wave speed, and beginning the integration at the local minimum to the left of the spectral peak, we obtain the result $D_{\infty}^{(2)} \simeq 1300\mathrm{cm}^2/\mathrm{s}$.

If the tidal peak (which is absent in many ocean regions) is neglected, observed spectra of BIG waves are in good agreement with theoretical spectra suggested in (Glazman, 1996 b). In particular, the frequency spectrum of BIG waves is given in scaled form (29) by

$$s(x) = a\frac{(x^4+7)(x^2+1)}{(x^2)^{1/3}(x^4-1)^{5/3}}, \tag{33}$$

valid for $x > 1$, where a. is the normalization required by (29). This form is actually not formalizable as its integral diverges at $x = 1$, a result also due to the breakdown of the f-plane approximation. As explained earlier, we take $s(x) \equiv 0$ for $x \leq 1 + \delta$, for various values of δ (see below). From the theory of BIG wave turbulence (Glazman, 1996 b), the prefactor u_0^2/f_0 may also be written in the form

$$u_0^2/f_0 = \frac{\alpha_4}{6a_0} R^{4/3}\epsilon^{1/3} \tag{34}$$

where ϵ is again the flux of wave energy (or, equivalently, the energy 'dissipation rate normalized by the fluid mass density) $R = c/f$ is the Rossby radius of deformation, and α_4 is a dimensionless coefficient analogous to the Kolmogorov constant in fluid turbulence. We then obtain $D_{\infty}^{(2)}$ in the form

$$D_{\infty}^{(2)} = B\epsilon^{2/3}R^{5/3}c_0^{-1} \tag{35}$$

where the dimensionless coefficient is given by

$$B = \frac{\pi\alpha_4}{144}\int_{1+\delta}^{\infty} dx \frac{(x^4+7)^2(x^2+1)^2(x^2-1)}{x^{10/3}(x^4-1)^{10/3}}. \tag{36}$$

Assuming $\alpha_4 \simeq 1$, numerical integration yields $B \simeq 0.8$ for $\delta = 0.1$. For the case $c = 3\mathrm{m/s}$, $R = 30\mathrm{km}$, and $\epsilon \sim 10^{-14}\,\mathrm{m}^2/\mathrm{s}^3$ (relevant to the mid-latitude ocean region employed in Fig. 1) one finds $D_{\infty}^{(2)} \simeq 34\mathrm{cm}^2/\mathrm{s}$, in rough agreement with the estimate above using the experimental spectrum (though the latter contains the additional tidal peak).

The fact that the Rossby radius increases as the latitude decreases - exceeding 200km near the equator – indicates a possible role of BIG waves as a factor in horizontal transport in tropical regions. In contrast, to the meso-scale and large-scale eddies which can propagate only westward, BIG waves can move in both westward and eastward directions and are hence capable of eastwardly transporting the various tracers in these regions.

VI. APPLICATIONS TOWIND-GENERATEDSURFACEGRAVITYWAVES

Application of the theory to the case of wind-generated surface gravity waves on deep water requires knowledge of the corresponding spectrum of water particle horizontal velocities given the measured spectrum of surface height variations, $S_\eta(\omega)$. This relationship depends on the depth, z, below the surface and is given by

$$S(\omega; z) = 2\omega^2 S_\eta(\omega) e^{-2\omega^2 z/g}. \tag{37}$$

Theoretical (Zakharov and Filonenko, 1966; Phillips, 1977; Phillips, 1985) and empirical (Glazman, 1994) studies yield a family of surface height spectra which maybe represented in the general form

$S_\eta(\omega)$ = **pg2(u/g)4’u-5+4’o(u/we),** *(38)*

where U is the mean wind speed above the sea surface, g is the acceleration due to gravity, $\omega_0 = g/U\xi$ is the spectral peak frequency, ξ is the *wave age* defined as the ratio of the phase speed of the waves at the spectral peak to the wind velocity U, $\Theta(\omega/\omega_0)$ is a smoothed step (Heaviside) function which imposes a smooth cutoff at frequencies below the spectral peak, and β is a dimensionless Phillips constant which is a slowly decreasing function of the wave age. Equation (38) with $\mu = 1/4$ reduces to the Zakharov-Filonenko spectrum which is controlled by the direct inertial cascade of energy toward smaller scales (analogous to the usual Kolmogorov cascade in isotropic turbulence), and is then valid at scales smaller than the driving scale. On the other hand the choice $\mu = 1/3$ corresponds to the inverse cascade of wave action (Zakharov and Zaslavskii, 1982) and is valid at scales larger than the driving scale, including the high frequency side of the spectral peak. In general, in the absence of unambiguous inertial ranges, the exponent $\mu = \mu(\xi)$ may be viewed as an effective exponent that generally increases with wave age (Glazman, 1994; Glazman, *et aL*, 1996). Notice that because the wavelengths of interest are in the range l-300m, Coriolis effects are negligible and no latitude dependence occurs in (38)

In order to omit effects of smaller scale ripples influenced by surface tension and other extraneous factors, a high frequency cutoff can be imposed in (38), thus forcing an exponential decay at frequencies above those associated with the '(intrinsic microscale" of the surface gravity range (Glazman and Weichman, 1989). However, for a finite depth z, the velocity spectrum (37) experiences a sufficiently fast high frequency roll-off to make the use of an intrinsic microscale unnecessary. Equations (37), (38) and the gravity wave dispersion law $\omega^2 = gk$ allow one to estimate the diffusion coefficient based on (28). For a typical open ocean case where $U = 10\text{m/s}$ and $\xi = 2$, one finds the dependence of the diffusion constant on depth as shown in Fig. 2. We conclude from this figure that within a few meters of the ocean surface, surface gravity wave induced diffusion is of the same order of magnitude as that caused by baroclinic inertia-gravity waves. However below 10m depths the effects of surface gravity waves are negligible.

Wave-induced turbulent transport in the immediate sub-surface layer is of great geophysical importance because the surface inhibits the growth of three-dimensional eddies which, otherwise, would give rise to classical eddy-induced diffusion. Therefore, the only mechanism competing with wave-induced diffusion in this region is molecular diffusion. While the above estimate for $D_{\infty}^{(2)}$ pertains to horizontal diffusion, the general theory we have developed is equally valid for the vertical component of the full three-dimensional diffusion coefficient tensor. Computation of this component would allow one to properly formulate the problem of turbulent transport *through* the ocean surface. This will be pursued in a later publication.

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FIGURES

FIG. 1. Frequency spectrum of ocean currents measured at 34.9°N, 55°W in the North Atlantic at a depth of 600m. The local ocean depth is 5506m. Units of frequency are cycles per hour (cph). The figure has been replotted based on Fig. 2 of (Fu, 1981).

FIG. 2. Diffusion constant due to surface gravity waves near the ocean surface with wind speed $U = 10\mathrm{m/s}$ and wave age $\xi = 2$.

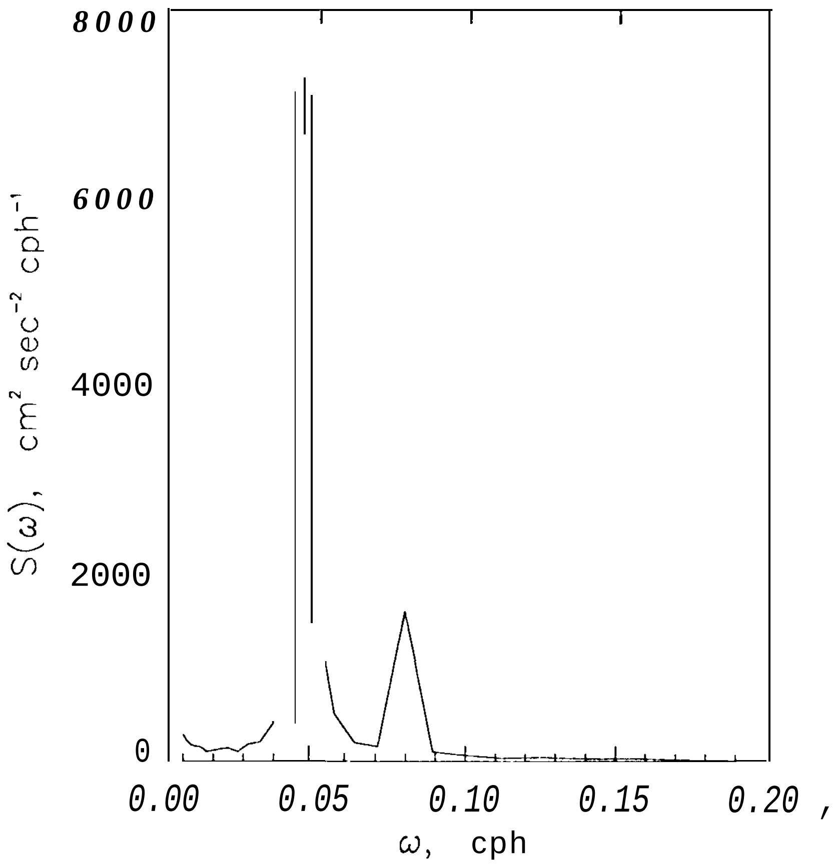
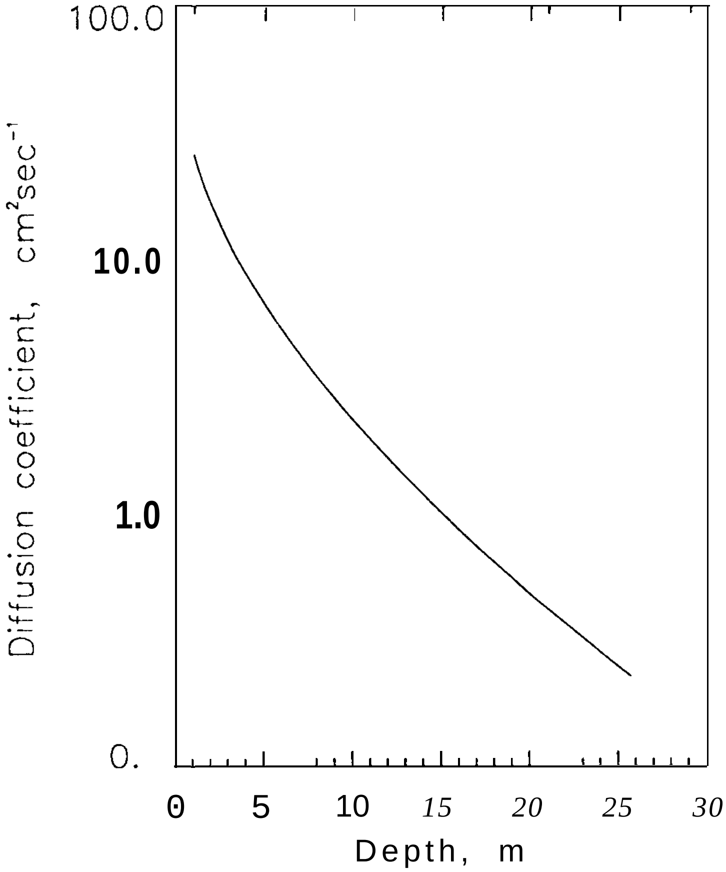


Fig. 1



100.0
10.0
1.0
0.
Diffusion coefficient, cm²sec⁻¹
0
5
10
15
20
25
30
Depth, m

Fig. 2